Finite rotations and angular velocity

Asher Peres
Department of Physics, Technion-Israel Institute of Technology, Haifa, Israel
(Received 5 January 1979; accepted 1 April 1979)

The angular velocity vector is not the time derivative of the vector which represents a finite rotation. The relationship between the two is derived explicitly.

The formula for rotating a vector $\mathbf{r}$ through a finite angle $\boldsymbol{\alpha}$ is well known.[1,2] An elegant notation was recently proposed by Koehler and Trickey.[3] The aim of this paper is twofold: to give a concise derivation of this formula and to obtain the relationship between the angular velocity $\boldsymbol{\omega}$ and the time derivative of $\boldsymbol{\alpha}$. These two are not the same (unless $\boldsymbol{\alpha}$ and $\dot{\boldsymbol{\alpha}}$ are parallel) because finite rotations do not commute, even with infinitesimal ones. The expression

$$d\boldsymbol{\alpha} = \boldsymbol{\alpha}(t+dt) - \boldsymbol{\alpha}(t)$$

is mathematically well defined, but is *not* the infinitesimal rotation required to carry $\mathbf{r}(t)$ into $\mathbf{r}(t+dt)$. The latter is the infinitesimal rotation $\boldsymbol{\omega}dt$. Therefore, $\boldsymbol{\omega} \neq \dot{\boldsymbol{\alpha}}$.

To obtain the formula[4] for a rotation through $\boldsymbol{\alpha}$, we divide it into a very large number N of infinitesimal rotations through an angle $\boldsymbol{\alpha}/N$. For the kth step, we have

$$\mathbf{r}_k = [1 + (\boldsymbol{\alpha}/N)\times] r_{k-1},$$

and therefore after N steps

$$\mathbf{r}_N = [1 + (\boldsymbol{\alpha}/N)\times]^N \mathbf{r}_0.$$

For $N \to \infty$, this gives

$$\mathbf{r} = e^{\boldsymbol{\alpha}\times}\mathbf{r}_0, \tag{1}$$

where $e^{\boldsymbol{\alpha}\times}$ means $\Sigma(\boldsymbol{\alpha}\times)^n/n!$, and

$$(\boldsymbol{\alpha}\times)^n \equiv \{\boldsymbol{\alpha}\times[\boldsymbol{\alpha}\times(\boldsymbol{\alpha}\times\cdots)]\}, \quad (n \text{ times}).$$

By virtue of

$$(\boldsymbol{\alpha}\times)^3 = -\alpha^2(\boldsymbol{\alpha}\times),$$

we can rewrite[3] Eq. (1) as

$$\mathbf{r} = \mathbf{r}_0 + \frac{\sin\alpha}{\alpha}(\alpha \times \mathbf{r}_0) + \frac{1-\cos\alpha}{\alpha^2}[\alpha \times (\alpha \times \mathbf{r}_0)],$$

which is the form best suited for practical calculations.

This can also be written as

$$\mathbf{r} = S\mathbf{r}_0,$$

where S is the orthogonal matrix with elements

$$S_{ik} = \cos\alpha\,\delta_{ik} + \frac{\sin\alpha}{\alpha}\epsilon_{ijk}\,\alpha_j + \frac{1-\cos\alpha}{\alpha^2}\alpha_i\alpha_k. \tag{2}$$

Now suppose that $\boldsymbol{\alpha}$ is a function of time. We then have

$$\dot{\mathbf{r}} = \dot{S}\ \mathbf{r}_0 = \dot{S}S^{-1}\,\mathbf{r}.$$

Since S is orthogonal, $\dot{S}S^{-1} = \dot{S}S^T$ is an antisymmetric matrix, which we shall call Ω. We can then write

$$\dot{\mathbf{r}} = \Omega\ \mathbf{r} = \boldsymbol{\omega}\times\mathbf{r},$$

where $\boldsymbol{\omega}$ is defined by

$$\epsilon_{ijk}\,\omega_j = \Omega_{ik} = \dot{S}_{ij}S_{kj}. \tag{3}$$

The explicit evaluation of this expression is quite tedious. It is more instructive to proceed as follows:

Any three-dimensional vector can be represented by a 3×3 antisymmetric matrix according to the rule $\alpha_j \to A_{ik} = \epsilon_{ijk}\alpha_j$. An expression like $\boldsymbol{\alpha}\times\mathbf{r}$ then becomes $A\mathbf{r}$ (we still represent $\mathbf{r}$ as a vector, not a matrix) and likewise $(\boldsymbol{\alpha}\times)^n\mathbf{r}$ becomes $A^n\mathbf{r}$. Thus Eq. (1) simply is $\mathbf{r} = e^A\mathbf{r}_0$, whence we obtain $S = e^A$. It follows that

$$\Omega = \frac{d}{dt}(e^A)e^{-A},$$

which can be expanded (see appendix)[5]

$$\Omega = \dot{A} + (1/2!)[A,\dot{A}] + (1/3!)[A,[A,\dot{A}]] + \cdots \tag{4}$$

We must now evaluate the commutator of two antisymmetric matrices. This is also an antisymmetric matrix and it can easily be checked that if A represents a vector $\boldsymbol{\alpha}$ and B a vector β, then $[A,B]$ represents the vector $\alpha\times\beta$. Therefore, Eq. (4) can be rewritten in vector notation as

$$\begin{aligned}\boldsymbol{\omega} &= \dot{\boldsymbol{\alpha}} + (1/2!)(\boldsymbol{\alpha}\times\dot{\boldsymbol{\alpha}}) + (1/3!)[\boldsymbol{\alpha}\times(\boldsymbol{\alpha}\times\dot{\boldsymbol{\alpha}})] + \cdots \\ &= \sum\frac{(\boldsymbol{\alpha}\times)^n\dot{\boldsymbol{\alpha}}}{(n+1)!} \\ &= \dot{\alpha} + \frac{1-\cos\alpha}{\alpha^2}(\alpha\times\dot{\alpha}) + \frac{\alpha-\sin\alpha}{\alpha^3}[\alpha\times(\alpha\times\dot{\alpha})],\end{aligned}$$

which is the desired result. As a check, this expression was also obtained from Eqs. (2) and (3), but with considerably more labor.

APPENDIX

To prove Eq. (4), let us define

$$\Omega(\lambda) = \frac{\partial}{\partial t}(e^{\lambda A})e^{-\lambda A},$$

where λ is a parameter independent of t. Obviously, $\Omega(0) = 0$ and

$$\frac{\partial\Omega(\lambda)}{\partial\lambda} = \dot{A} + [A,\Omega(\lambda)]. \tag{5}$$

We can thereby obtain $\Omega(\lambda)$ iteratively, as a power series in λ:

$$\Omega(\lambda) = \lambda\dot{A} + \frac{\lambda^2}{2!}[A,\dot{A}] + \frac{\lambda^3}{3!}[A,[A,\dot{A}]] + \cdots$$

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ndeed, it is readily checked that the rhs vanishes for $\lambda =$) and satisfies the differential equation (5). Finally, setting $\lambda = 1$ gives the desired result.

[1]E. T. Whittaker, *Analytical Dynamics of Particles and Rigid Bodies*, 4th ed. (Cambridge University, Cambridge, England, 1937), p. 6.
[2]H. C. Corben and P. Stehle, *Classical mechanics*, 2nd ed. (Wiley, Nev York, 1960), p. 374.
[3]T. R. Koehler and S. B. Trickey, Am. J. Phys. **46**, 650 (1978).
[4]Our angle α is the same as θ of Ref. 2 and $-\theta$ of Ref. 3.
[5]A. Messiah, *Quantum mechanics*, 1st ed. (North-Holland, Amsterdam 1961), Vol. 1, p. 340.